§11. dimension theory

§10.1 Hilbert functions

- $A = \bigoplus_{n=0}^{\infty} A_n$ noeth. graded ring.

  $(10.7) \Rightarrow \begin{cases} A_0 = \text{noeth. ring} \\ A = A_0\text{-alg generated by some } x_1, \cdots, x_i, \cdots, x_s \end{cases}$

  homo. of deg. $k_i > 0$

- $M = \bigoplus_{n=0}^{\infty} M_n$ f.g. graded A-module.

  Fact: M_n = f.g. A_0-module.

  pf: M generated by $m_1, \cdots, m_j, \cdots, m_t$ homogeneous of deg. r_j

  $\Rightarrow \forall x \in M_n,\ x = \sum_{j=1}^{t} f_j(x) \cdot m_j,\quad f_j(x) = \sum_{i=0}^{\infty} f_j^{(i)}(x) \in A$

  $\Rightarrow x = \sum_{j=1}^{t} f_j^{(n-r_j)}(x) \cdot m_j$ → 拆成单项式.

  $\Rightarrow M$ is generated by all

  $\{ g_j(x) m_j \mid g_j(x) \text{ monomial in } x_i \text{ of deg } n-r_j \}$

- λ = additive function (with values in $\mathbb{Z}$) on the class of all f.g. A_0-module.

  i.e. $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow \lambda(M) = \lambda(M') + \lambda(M'')$.

  example: A_0 = Artin, $\lambda(M) = \ell(M)$ length of M.

- Poincaré series of M (w.r.t. λ) is

$$P(M,t) := \sum_{n=0}^{\infty} \lambda(M_n)\, t^n \in \mathbb{Z}[[t]].$$

Thm 11.1 (Hilbert, Serre) $\exists\, f(t) \in \mathbb{Z}[t]$ & $k_1, \cdots, k_s \geq 1$ s.t.

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s} (1-t^{k_i})}$$

Pf: Induction on s. $(A = A_0[x_1, \cdots, x_s])$

1) $s=0 \Rightarrow A = A_0$, $M =$ f.g. A_0-module $\Big\} \Rightarrow M_n = 0$, for $n \gg 0$.

$\Rightarrow P(M,t) \in \mathbb{Z}[t]$.

2) Suppose $s > 0$ & thm holds for $s-1$.

- $$0 \to \underset{\text{kernel}(x_s)}{K_n} \to M_n \xrightarrow{x_s} M_{n+k_s} \to \underset{\text{coker}(x_s)}{L_{n+k_s}} \to 0 \qquad (*)$$

$K := \bigoplus_n K_n$ f.g. A-mod $(\Leftarrow K \subseteq M)$

$L := \bigoplus_n L_n$ f.g. A-mod $(\Leftarrow M \twoheadrightarrow L)$

- $x_s \cdot K = 0 = x_s \cdot L$

$\Rightarrow K$ and L = f.g. $A/(x_s) = A_0[\bar{x}_1, \cdots, \bar{x}_{s-1}]$ - mod.

$\overset{\text{induction}}{\Longrightarrow}$ $P(K,t)$ & $P(L,t)$ are of the form in thm

• $(*) \Rightarrow \lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$

t^{n+k_s}

$$\Rightarrow t^{k_s}\sum_{n=0}^{\infty}\lambda(K_n)t^n - t^{k_s}\sum_{n=0}^{\infty}\lambda(M_n)t^n + \sum_{n=0}^{\infty}t^{n+k_s}M_{n+k_s} - \sum_{n=0}^{\infty}t^{n+k_s}\lambda(L_{n+k_s}) = 0$$

$$\Rightarrow (1-t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t)$$

Poly.

$\Rightarrow$ ✓

Def: $d(M) :=$ order of the pole of $P(M,t)$

Cor 11.2: $k_1 = \cdots = k_s = 1$

$\Rightarrow \exists\, h(t) \in \mathbb{Q}[t]$ of deg. $d-1$ s.t.

$$\lambda(M_n) = h(n) \quad \text{for } n \gg 0.$$

h is called the Hilbert function (or poly) of M w.r.t. λ

Pf: $\lambda(M_n) = $ coefficient of t^n in $f(t)\,(1-t)^{-d}$

$= $ coeff. of t^n in $\sum_{k=0}^{N} a_k t^k \cdot \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^k$

$\overset{n \geq N}{=} \sum_{k=0}^{N} a_k \binom{d+n-k-1}{d-1}$

leading term

$$\left(\sum_{k=0}^{N} a_k\right) \frac{n^{d-1}}{(d-1)!} = f(1)\frac{n^{d-1}}{(d-1)!} \neq 0 .$$

Prop 11.3. If $x \in A_k$ is not zero-divisor in $M^{\neq 0}$, then

$$d(M/xM) = d(M) - 1$$

Pf: $0 \to M_n \xhookrightarrow{x} M_{n+k} \to M_{n+k}/xM_n \to 0$

$\Rightarrow (1-t^k)\,P(M,t) = P(M/xM, t) + \underbrace{g(t)}_{\text{poly.}}$ ☐

$M = 0 \Rightarrow d(M) = -\infty = d(M/xM) \Rightarrow \checkmark$

$M \neq 0 \Rightarrow \exists\, n_0$ s.t. $M_{n_0} \neq 0 \Rightarrow M_{nk+n_0} \supseteq x^n M_{n_0} \neq 0$

$\Rightarrow P(M,t) \neq$ polynomial

$\Rightarrow P(M,t)$ has pole $\Rightarrow d(M) \geq 1$.

$\Rightarrow (1-t^k)\,P(M,t)$ has pole of deg $d(M)-1$ at 1

$\Rightarrow d(M/xM) = d(M) - 1$ ☐

Artin.

Example: $A = A_0[x_1, \dots, x_s]$, $\deg x_i = 1$.

$(\lambda(M) = \ell(M)$ & $\lambda_0 := \ell(A_0) \neq 0$

$$\Rightarrow P(A,t) = \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} \cdot \lambda_0 \cdot t^n = \frac{\lambda_0}{(1-t)^s}$$

pf: $\exists \binom{s+n-1}{s-1}$ monomials of deg n □

Prop 11.4
- $(A, \mathfrak{m})$ = noeth. local ring.
- $\mathfrak{q}$ = $\mathfrak{m}$-primary
- M = f.g. A-module
- (M_n) = stable $\mathfrak{q}$-filtration of M. Then

i) $\ell(M/M_n) < \infty \quad \forall n \geq 0$.

ii) $\exists$ polynomial $g(n)$ of deg $\leq s$ s.t.

$$\ell(M/M_n) = g(n) \qquad n >> 0.$$

where s is the least number of generators of $\mathfrak{q}$

iii) Leading term of $g(n)$ depends only on M and $\mathfrak{q}$ not on (M_n).

Pf: i) $\cdot$ A = noeth.

$$\overset{(8.5)}{\Rightarrow} \quad G(A) := \bigoplus_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1} \quad = \text{noeth.}$$

$\cdot$ M = f.g. A-module

$$\overset{(10.22)}{\Rightarrow} \quad G(M) = \bigoplus_{n=0}^{\infty} M_n/M_{n+1} = \text{f.g. } G(A)\text{-module}$$

$$\Rightarrow \quad M_n/M_{n+1} = \text{f.g. } A_0\text{-module.}$$

$$\overset{A_0 = \text{Artin}}{\Rightarrow} \quad \ell(M_n/M_{n+1}) < \infty$$

$$\Rightarrow \quad \ell(M/M_n) = \sum_{r=1}^{n} \ell(M_{r-1}/M_r) < \infty.$$

ii) $(11.2) \Rightarrow \exists$ poly $f(n)$ of deg $\leq s-1$ s.t.

$$f(n) = \ell(M_n/M_{n+1}) \quad n \gg 0.$$

$$\Rightarrow \quad \ell_{n+1} - \ell_n = f(n) \qquad n \gg 0.$$

$$\Rightarrow \quad \exists \quad g(n) = \text{poly of deg} \leq s.$$

$$\ell_n = \ell_N + \underbrace{f(N+1) + f(N+2) + \cdots + f(n)}$$

$$f = \sum_{k=0}^{d} a_k n^k \quad \Rightarrow \quad \sum_{k=0}^{d} a_k \Big(\underbrace{(N+1)^k + (N+2)^k + \cdots + n^k}\Big)$$

poly of n of deg $k+1$.

iii) $(\tilde{M}_n)$ another $\mathfrak{q}$-filtration.

$$\overset{(10.6)}{\Rightarrow} \exists N \text{ s.t. } \begin{cases} \tilde{M}_{n+N} \subseteq M_n \\ M_{n+N} \subseteq \tilde{M}_n \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{g}(n+N) \geq g(n) \\ g(n+N) \geq \tilde{g}(n) \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\tilde{g}(n)}{g(n)} = 1 \Rightarrow \checkmark \qquad \square$$

Notion: 1) poly $g(n)$ corr. to $(\mathfrak{q}^n M)$ is denote by $\chi_{\mathfrak{q}}^M(n)$.

$$\chi_{\mathfrak{q}}^M(n) := \ell(M/\mathfrak{q}^n M) \qquad \underline{n \gg 0}$$

2) $\chi_{\mathfrak{q}}(n) := \chi_{\mathfrak{q}}^A(n)$ characteristic polynomial of the $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

Prop 11.6. $\deg \chi_{\mathfrak{q}}(n) = \deg \chi_{\mathfrak{m}}(n)$

pf: $\mathfrak{m} \supseteq \mathfrak{q} \supseteq \mathfrak{m}^r \Rightarrow \mathfrak{m}^n \supseteq \mathfrak{q}^n \supseteq \mathfrak{m}^{nr}$

$$\Rightarrow \chi_{\mathfrak{m}}(n) \leqslant \chi_{\mathfrak{q}}(n) \leqslant \chi_{\mathfrak{m}}(rn) \qquad \square$$

Notion: $d(A) := \deg \chi_{\mathfrak{q}}(n)$.

Fact: $d(A) = d(G_{\mathfrak{m}}(A))$

§11.2 dimension theory of noetherian local rings

(A, m) = noeth. local ring

- $\delta(A)$:= least number of generators of an m-primary ideal of A

$d(A) := \deg \chi_q(n)$ $\left(\ell(A/q^n) = O(n^{d(A)})\right)$

$\dim A$:= max. length $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_r = m$

aim of this section:

$$\delta(A) \geq \delta(A) \geq \dim(A) \geq \delta(A)$$

Prop 11.7 $\delta(A) \geq d(A)$.

Pf: (11.5) + (11.6) $\square$

Prop 11.8 M = f.g. A-module. Let $x \in A$ be a non-zero-divisor in M. Then

$$\deg \chi_q^{M/xM} = \deg \chi_q^M - 1$$

Pf: $N := xM \subseteq M \Rightarrow M \xrightarrow{\sim} N$ (as A-mods)
$m \mapsto xm$

$\left.\begin{array}{l} M' := M/xM \\ N_n := N \cap \mathfrak{q}^n M \end{array}\right\}$

$\Rightarrow 0 \to N/N_n \to M/\mathfrak{q}^n M \to M'/\mathfrak{q}^n M' \to 0$

$\xrightarrow{g(n) := \ell(N/N_n) \quad n \gg 0} g(n) - \chi_{\mathfrak{q}}^{M}(n) + \chi_{\mathfrak{q}}^{M'}(n) = 0 \quad n \gg 0$

Artin-Rees (10.9) $\Rightarrow (N_n)$ = stable $\mathfrak{q}$-filtration

$\Rightarrow g(n)$ & $\chi_{\mathfrak{q}}^{M}(n)$ has the same Leading term

$\Rightarrow \deg \chi_{\mathfrak{q}}^{M'} < \deg \chi_{\mathfrak{q}}^{M}(n) = \deg g(n)$ $\square$

Cor 11.9. A = noeth. local $x \neq$ zero divisor in A. Then

$$d(A/(x)) \leq d(A) - 1$$

Prop 11.10. $d(A) \geq \dim A$.

Pf: Induction on $d = d(A)$.

$d=0 \Rightarrow \chi_m(n) = \text{const}$.

$\Rightarrow \ell(A/m^n) = \text{constastant} \quad n \gg 0$.

$\Rightarrow m^n = m^{n+1} \quad n \gg 0$

$\overset{(2.6)}{\Rightarrow} m^n = 0$

$\Rightarrow A = \text{Artin}$ & $\dim A = 0$. ✓

Suppose $d > 0$. $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ any chain

1°. $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_0 \Rightarrow x' = x + \mathfrak{p}_0 \in A' = A/\mathfrak{p}_0$

integral

$\overset{(11.9)}{\Rightarrow} d(A'/(x')) \leq d(A') - 1$

2°. $m' = \text{max ideal of } A'$

$\Rightarrow A \overset{\pi}{\twoheadrightarrow} A'$

$\Rightarrow \pi(m) \subseteq m'$

$$\Rightarrow \quad A/\mathfrak{m}^n \twoheadrightarrow A'/\mathfrak{m}'^n$$

$$\Rightarrow \quad \ell(A/\mathfrak{m}^n) \geq \ell(A'/\mathfrak{m}'^n)$$

$$\Rightarrow \quad d(A) \geq d(A')$$

$$\overset{1^\circ}{\Rightarrow} \quad d(A'/(x')) \leq d-1$$

$$\overset{\text{induction}}{\Rightarrow} \quad r-1 \leq \dim(A'/(x')) \leq d(A'/(x')) \leq d-1$$

$$\Rightarrow \quad r \leq d \qquad \square$$

Cor 11.11 A = noeth. local ring $\Rightarrow$ $\dim A < \infty$.

height of prime ideal $\mathfrak{p}$

$$ht(\mathfrak{p}) := \dim(A_{\mathfrak{p}})$$

$\overset{(3.13)}{=}$ the supremum of chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$$

Cor 11.12 In a noeth. ring,

i) every prime ideal has finite height

ii) the set of prime ideals satisfies DCC. (Desending chain Condition)

Rmk: $depth(\mathfrak{p}) := \dim(A/\mathfrak{p})$ could be ∞.

Prop 11.13. (A, m) = noeth. local of dim d. Then there exists an m-primary ideal in A generated by d elements $x_1, \cdots, x_d$. Therefore $\dim A \geq \delta(A)$.

Pf: Construct $x_1 \cdots x_d$ inductively s.t.

$$\forall \text{ prime } \mathfrak{p} \supseteq \{x_1, \cdots, x_i\} \Rightarrow ht(\mathfrak{p}) \geq i. \qquad (*)$$

Suppose $i > 0$ & $x_1, \cdots, x_{i-1}$ are constructed.

$I = (x_1, \cdots, x_{i-1}) \triangleleft A$

$\Rightarrow \Sigma = \{\mathfrak{p} : \text{minimal prime ideals of } I\}$

$= \{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s, \cdots, \mathfrak{p}_t\} \quad (ht \geq i-1)$

assume $ht(\mathfrak{p}_1) = \cdots = ht(\mathfrak{p}_s) = i-1$

$ht(\mathfrak{p}_l) \geq i \quad \forall s < l \leq t.$

1° $s = 0$. i.e. $ht(\mathfrak{p}_l) \geq i \quad \forall 1 \leq l \leq t$

$\Rightarrow \forall x_i \in m$, $\{x_1, \cdots, x_i\}$ satisfies $(*)$.

2° $s \neq 0$.

$$\overset{(1.11)}{\Rightarrow} m \neq \bigcup_{j=1}^{s} \mathfrak{p}_j \quad \Big(\overset{\text{not maximal}}{\mathfrak{p}_j \neq m} \ \forall j = 1, \cdots, s\Big)$$

$\Rightarrow \forall x_i \in m \setminus \bigcup_{j=1}^{s} \mathfrak{p}_j$

$\forall$ prime ideal $\mathfrak{q} \supseteq (x_1, \cdots, x_i) \supsetneq (x_1, \cdots, x_{i-1})$

$\Rightarrow \exists\ \bar{j} \in \{1, \cdots, r\}$ s.t.

$\mathfrak{q} \supseteq \mathfrak{p}_{\bar{j}}$

1': $\bar{j} \in \{1, \cdots, s\} \Rightarrow x_i \notin \mathfrak{p}_{\bar{j}} \Rightarrow \mathfrak{q} \supsetneq \mathfrak{p}_{\bar{j}}$

$\Rightarrow ht(\mathfrak{q}) \geq ht(\mathfrak{p}_{\bar{j}}) + 1 = i$

2': $\bar{j} \notin \{1, \cdots, s\} \Rightarrow ht(\mathfrak{q}) \geq ht(\mathfrak{p}_{\bar{j}}) \geq i$.

$\Rightarrow$ (*) holds for $\{x_1, \cdots, x_i\}$.

Consider $(x_1, \cdots, x_d)$.

$\forall\ \mathfrak{p}$ = prime ideal of $(x_1, \cdots, x_d)$

$\Rightarrow ht(\mathfrak{p}) \geq d \Rightarrow \mathfrak{p} = m$

$\Rightarrow (x_1, \cdots, x_d) = m$-primary. □

Thm 11.14 (dimension thm) (A, m) = noeth. local.

$$\delta(A) = d(A) = \dim A.$$

pf: (11.7), (11.10), (11.13) □.

Example: $A = k[x_1, \cdots, x_n]_m \qquad m = (x_1, \cdots, x_n)$.

$\Rightarrow G_m(A) \cong k[\bar{x}_1 \cdots \bar{x}_n]$

$\Rightarrow P(G_m(A), t) = (1-t)^{-n}$

$\Rightarrow \dim A = n$.

Cor 11.15 : $\dim A \leq \dim_k(m/m^2)$.

pf: $m/m^2 = \bigoplus_{i=1}^{s} k \cdot \bar{x}_i \Rightarrow m = \sum_{i=1}^{s} A x_i$ $\square$

Cor. 11.16: A = noeth. $x_1, \cdots, x_r \in A$.

$\mathfrak{p}$ = a minimal ideal belonging to $(x_1, \cdots, x_r)$.

Pf: $(x_1 \cdots x_r) A_{\mathfrak{p}} \triangleleft A_{\mathfrak{p}}$ is $\mathfrak{p} A_{\mathfrak{p}}$-primary.

$\Rightarrow ht(\mathfrak{p}) = \dim A_{\mathfrak{p}} \leq r$ $\square$

Cor 11.17 (Krull's principal ideal thm) A = noeth. $x \in A \setminus A^\times$

$x \neq$ zero divisor $\Rightarrow ht(\mathfrak{p}) = 1$, $\forall \mathfrak{p}$ = mini. prim ideal of (x).

Pf: $(11.16) \Rightarrow ht(\mathfrak{p}) \leq 1$.

Suppose $ht(\mathfrak{p}) = 0 \Rightarrow \mathfrak{p}$ belong to $(0) \overset{(4.7)}{\Rightarrow} x \in \mathfrak{p}$ zero divisor ↯.

Cor 11.18 (A, m) = noeth. local, $x \in m$, non-zero-divisor. Then

$$\dim A/(x) = \dim A - 1$$

Pf: $d := \dim A/(x)$.

- $d \overset{(11.14)}{=} d(A/(x)) \overset{(11.9)}{\leq} d(A) - 1 \overset{11.14}{=} \dim(A) - 1$
- $\forall\, x_1, \cdots, x_d \in m$ s.t.

$$(\bar{x}_1, \cdots, \bar{x}_d) = m/(x)\text{-primary}$$

$\Rightarrow (x, x_1, \cdots, x_d) = m$-primary.

$\Rightarrow \dim A \leq d+1$ $\square$

Cor 11.19: $\hat{A}$ = m-adic completion of A. Then

$$\dim A = \dim \hat{A}$$

Pf: $(10.15) \Rightarrow A/m^n \cong \hat{A}/\hat{m}^n \Rightarrow \chi_m(n) = \chi_{\hat{m}}(n)$ $\square$

$\{x_1 \cdots x_d\}$ is called a <u>system of parameters</u> for A, if

$$\dim A = d \ \ \& \ \ (x_1, \cdots, x_d) = m\text{-primary}.$$

Prop 11.20: • $x_1, \dots, x_d$ = system of parameters for A.

$\mathfrak{q} = (x_1, \dots, x_d)$,

• $f \in A[t_1, \dots, t_d]$ homog. of deg s. Then

$$f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1} \Rightarrow \text{all coeff. of } f \in \mathfrak{m}$$

Pf: • Consider $\alpha : (A/\mathfrak{q})[t_1, \dots, t_d] \twoheadrightarrow G_{\mathfrak{q}}(A)$ hom. of graded rings

$t_i \mapsto \bar{x}_i$

$$\alpha(\bar{f}) = \bar{f}(\bar{x}_1, \dots, \bar{x}_d) = 0 \in \mathfrak{q}^s/\mathfrak{q}^{s+1}$$

$$\Rightarrow \quad (A/\mathfrak{q})[t_1, \dots, t_d]/(\bar{f}) \twoheadrightarrow G_{\mathfrak{q}}(A)$$

• Suppose some coeff. of f is a unit.

$\Rightarrow$ $\bar{f} \neq$ zero-divisor in $(A/\mathfrak{q})[t_1, \dots, t_d]$

$$\Rightarrow \quad d \overset{(11.14)}{=} d(G_{\mathfrak{q}}(A))$$

$$\leq d((A/\mathfrak{q})[t_1 \dots, t_d]/(f))$$

$$\overset{(11.3)}{=} d((A/\mathfrak{q})[t_1 \dots, t_d]) - 1$$

$$\overset{\text{Ex } 11.3}{=} d - 1 \qquad \lightning.$$

Cor 11.21 : Suppose $k \subsetneq A$ is a field s.t.

$$\begin{tikzcd} k \arrow[rr, "\cong"] \arrow[dr] & & A/\mathfrak{m} \\ & A \arrow[ur] & \end{tikzcd}$$

$(x_1, \cdots, x_d)$ = system of parameters. Then

$x_1, \cdots, x_d$ are algebraically independent over k.

Pf : Assume $f(x_1, \cdots, x_d) = 0$ with $f \in k[t_1, \cdots, t_d] \setminus \{0\}$.

$$f = \underbrace{f_s}_{\text{homog. of deg } s.} + \underbrace{\tilde{f}}_{\text{higher terms}}$$

$\Rightarrow f_s(x_1, \cdots, x_d) = -\tilde{f}(x_1, \cdots, x_d) \in \mathfrak{q}^{s+1}$

$\Rightarrow$ coefficients of $f_s \in \mathfrak{m}$

$\overset{\mathfrak{m} \cap k = 0}{\Rightarrow}$ coefficents of $f_s = 0$

$\Rightarrow f_s = 0$ $\lightning$. $\square$

§ 11.3 Regular local Rings.

Thm 11.22. $(A, \mathfrak{m}, k)$ = noeth. local of dim. d. TFAE.

i). $G_{\mathfrak{m}}(A) \cong k[t_1, \cdots, t_d]$ (as graded ring)

ii). $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$

iii) $\mathfrak{m}$ can be generated by d elements.

Pf: i) $\Rightarrow$ ii) clear. $\left(\mathfrak{m}/\mathfrak{m}^2 \cong \bigoplus_{i=1}^{d} k t_i .\right)$

ii) $\Rightarrow$ iii) Nakayama

iii) $\Rightarrow$ i) Let $\mathfrak{m} = (x_1, \cdots, x_d)$

$$\overset{(11.20)}{\Longrightarrow} \alpha : k[t_1, \cdots, t_d] \xrightarrow{\cong} G_{\mathfrak{m}}(A)$$
$$t_i \longmapsto \bar{x}_i \qquad \square$$

Def: regular local ring = noeth. local ring satisfying i), ii), iii). in Thm.

Lem 11.23: $\mathfrak{a} \triangleleft A$ s.t. $\bigcap_n \mathfrak{a}^n = 0$

$$G_{\mathfrak{a}}(A) = \text{int. domain} \Rightarrow A = \text{int. domain}.$$

Pf: $\forall x, y \in A \setminus \{0\} \Rightarrow x \in \mathfrak{a}^r \setminus \mathfrak{a}^{r+1}$ & $y \in \mathfrak{a}^s \setminus \mathfrak{a}^{s+1}$ for some r, s.

$\Rightarrow \bar{x}, \bar{y} \in G_{\mathfrak{a}}(A) \setminus \{0\}$

$\Rightarrow \quad \overline{xy} = \bar{x}\cdot\bar{y} \neq 0 \in \mathfrak{a}^{r+s}/\mathfrak{a}^{r+s+1} \subseteq G_{\mathfrak{a}}(A)$

$\Rightarrow \quad xy \notin \mathfrak{a}^{r+s+1}$

$\Rightarrow \quad xy \neq 0$ $\square$

Fact : 1) regular local ring of dim 1 $\overset{(9.2)}{=}$ DVR

2) A = local int. domain

$G_m(A)$ = Integrally closed $\Rightarrow$ A = Integral closed

3). regular $\Rightarrow$ Integrally closed.
$\not\Leftarrow$

Prop 11.24. A = noeth. local ring. Then

A = regular $\Leftrightarrow$ $\hat{A}$ = regular.

Pf : A = local $\overset{10.16}{\Longrightarrow}$ $\hat{A}$ = local

A = noeth. $\overset{10.26}{\Longrightarrow}$ $\hat{A}$ = noeth.

(11.19) $\Rightarrow$ $\dim A = \dim \hat{A} =: d$

A = regular $\overset{\text{Def}}{\Longleftrightarrow}$ $G_m(A) \cong k[t_1, \dots, t_d]$

$\Longleftrightarrow$ $G_{\hat{m}}(\hat{A}) \cong k[t_1, \dots, t_d]$

$\overset{\text{Def}}{\Longleftrightarrow}$ $\hat{A}$ = regular $\square$

§11.4 transcendental dimension

- $k = \bar{k}$ alg. closed field
- V = irr affine variety over k i.e., $\exists$ prime idea $\mathfrak{p} \lhd k[t_1, \cdots, t_n]$

$$V = \{ x \in \mathbb{A}^n \mid f(x) = 0 \quad \forall f \in \mathfrak{p} \}$$

$A(V) := k[t_1, t_2, \cdots, t_n]/\mathfrak{p}$ the coordinate ring of V.

$k(V) := \mathrm{Frac}(A(V))$ field of rational function on V.

- $\dim V := \mathrm{tran.deg}_k\, k(V)$.

Fact (Nullstellensatz) $V \xleftrightarrow{1:1} \{ m \lhd A(V) \mid m = \text{maximal} \}$

$$x = (x_1, \cdots, x_n) \mapsto m_x = (t_1 - x_1, t_2 - x_2, \cdots, t_n - x_n) \lhd A(V)$$

Thm 11.25. $\dim V = \dim A(V)_m \quad \forall m$ maximal

Pf: $d := \dim V$

$A = A(V) \Rightarrow \exists x_1, \cdots, x_d \in A$ s.t.

$B = k[x_1, \cdots, x_d] \hookrightarrow A$ is integral

$\overset{§5}{\Rightarrow} \dim A_m = \dim B_n = d$

<u>Cor 11.27</u> $\dim A(V) = \dim A(V)_m \quad \forall m$.

Pf: $\dim A(V) := \sup\limits_m \dim A(V)_m = \dim V = \dim A(V)_m$ ∎